

Guaranteed Accuracy for Conic Programming Problems in Vector Lattices

Christian Jansson

February 1, 2008

Institute for Reliable Computing, Technical University Hamburg–Harburg, Schwarzenbergstraße 95, 21071 Hamburg, Germany, e-mail: jansson@tu-harburg.de, Fax: ++49 40 428782489.

Keywords: Linear programming, semidefinite programming, conic programming, convex programming, combinatorial optimization, rounding errors, ill-posed problems, interval arithmetic, branch-bound-and-cut

AMS Subject classification: primary 90C25, secondary 65G30

Abstract

This paper presents rigorous forward error bounds for linear conic optimization problems. The error bounds are formulated in a quite general framework; the underlying vector spaces are not required to be finite-dimensional, and the convex cones defining the partial ordering are not required to be polyhedral. In the case of linear programming, second order cone programming, and semidefinite programming specialized formulas are deduced yielding guaranteed accuracy. All computed bounds are completely rigorous because all rounding errors due to floating point arithmetic are taken into account. Numerical results, applications and software for linear and semidefinite programming problems are described.

1 Introduction

In this paper forward error bounds for the optimal value of linear conic optimization problems as well as certificates of feasibility and infeasibility are presented, including the discussion of

rounding-off errors and details of implementation. These rigorous bounds aim to prove how accurate the approximate results computed by any conic solver are. The underlying vector spaces are in general infinite-dimensional, that is the bounds are developed in the framework of functional analysis.

Forward and backward error analysis together with a detailed discussion of rounding-off errors and condition numbers for matrix problems were first described in the outstanding papers published sixty years ago by von Neumann and Goldstine [32] and Turing [45]. Turing writes:

Error estimates can be of two kinds. We may wish to know how accurate a certain result is, and be willing to do some additional computation to find out. A different kind of estimate is required if we are planning calculations and wish to know whether a given method will lead to accurate results. In the former case we do not care what quantities the error is expressed in terms of, provided they are reasonably easily computed. With these estimates we wish to be absolutely sure that the error is within the range stated, but at the same time not to state a range which is very much larger than necessary. With the second type of estimate, the error is preferably expressed in terms of quantities whose meaning is sufficiently familiar that the general run of values involved may at least be guessed at.

Particularly, forward error bounds for the inverse of a matrix including a discussion of the effects of rounding-off errors are presented there. Today one would speak in this context of verified or rigorous error bounds, and thus these two papers can be viewed as the pioneering work in the field of verification methods, a part of numerical analysis. Forward error bounds are propagated in interval arithmetic; see the textbooks Alefeld and Herzberger [1], Moore [22], and Neumaier [27], [28]. But also in other areas the interest in rigorous forward error bounds is growing. Parlett [34], for example, remarks in relation to the numerical accuracy of eigenvalue problems:

For some of us, however, it has taken nearly 40 years to realize that backward stability is not enough.

Also Trefethen writes in [44] about the future of Numerical Analysis

I expect that most of the numerical computer programs of 2050 will be 99% intelligent and just 1% actual “algorithm” if such a distinction makes sense. Hardly anyone will know how they work, but they will be extraordinarily powerful and reliable, and will often deliver results of guaranteed accuracy.

Linear conic optimization refers to problems with a linear objective function and linear constraints where the variables are restricted to a cone. In general these problems are non-smooth. Linear programming, quadratically convex programming, second order cone programming and semidefinite programming are special cases. Since each convex problem can be described equivalently as a linear conic problem, the latter provides a universal form of convex programming (see Nemirovski [24], [25]). Thus not surprisingly, a large variety of applications of conic programming are known from areas like system and control theory, combinatorial optimization, signal processing and communications, machine learning, quantum chemistry, and many others. For an elaborate bibliography the reader is referred to Wolkowicz [47].

Nesterov and Nemirovski [25] have shown that self-concordant barriers apply to many conic problems yielding polynomial time interior point methods. Renegar [36] has investigated the sensitivity of infinite-dimensional conic optimization problems, and in [37] he analyzed interior point methods. He introduced a condition number for conic optimization, which is a generalization of the condition numbers defined by von Neumann, Goldstine, and Turing. This *condition number* is the scale-invariant reciprocal of the smallest data perturbation that will render the perturbed data instance either primal or dual infeasible. It is used in sensitivity analysis and moreover can be viewed as a problem instance size of conic optimization problems yielding important results in complexity theory. A problem is called *ill-posed* if this condition number is infinite, that is the distance to primal or dual infeasibility is zero.

One of Renegar's main results is that the sensitivity of the optimal solutions and the optimal value can be bounded by the condition, and especially he proved that the bounds for the optimal value depend cubically on the inverses of the relative distances to primal and dual infeasibility. Renegar shows that this bound cannot be improved in general. For an ill-posed problem this result means that there exist arbitrarily small perturbed data instances such that the difference between the optimal value of the original problem and the perturbed problem is arbitrarily large, but the optimality conditions for the perturbed problem almost coincide with the optimality conditions for the original problem. Since conic solvers are terminated when the optimality conditions are satisfied approximately it cannot be distinguished between the optimal values of the original and the perturbed problem in the case of ill-conditioned or ill-posed problems. A consequence is that the noise introduced by floating point arithmetic may occasionally yield to wrong termination and nonsensical computational results.

In this paper we show how certain weak boundedness qualifications on ε -optimal solutions can be used to compute rigorous forward error bounds for the exact optimal value, also for ill-conditioned or even for ill-posed problems. Such qualifications and even more restrictive assumptions, like certain smoothness properties, are customary when solving ill-posed problems with regularization methods. It need not to be assumed that Slater's constraint

qualifications are fulfilled. The rigorous error bounds provide more safety for conic optimization problems, and they provide rigorous results in branch-and-bound algorithms for global and combinatorial optimization problems. Another application are computer-assisted proofs where it is mandatory to control all rounding-off errors (see for example Neumaier [30] and Rump [41]). It should be made clear that we do not investigate regularization methods. In this paper we assume that approximations computed by some conic solver (with or without regularization) are given, and these approximations are then used for computing the error bounds. It is of particular importance that the computation of the error bounds can be done outside the code of any imaginable solver as a reliable postprocessing routine, providing a correct output for the given input. Especially, we show for some combinatorial problems how branch-and-bound algorithms can be made safe, even if ill-posed relaxations are used. Numerical results for some ill-posed and ill-conditioned problems are included.

Ill-conditioned and ill-posed problems are not rare in practice, they occur even in linear programming. Ordóñez and Freund [33] stated that 71% of the lp-instances in the NETLIB Linear Programming Library [26] are ill-posed. This library contains many industrial problems. Recently Freund, Ordóñez and Toh 2006 [8] have shown that 32 out of 85 problems of the SDPLIB are ill-posed.

The presented results in this paper formalize a viewpoint which apparently has not been made in conic programming. They can be viewed as an extension of methods for linear programming ([14] and Neumaier and Shcherbina [31]), and for smooth convex programming (see [13]) to ill-conditioned and ill-posed non-smooth problems using the framework of functional analysis.

The paper is organized as follows. After introducing some notation and basic definitions in Section 2, we consider in the next section conic optimization problems. Then in Section 4 verified lower and upper bounds of the optimal value in the infinite-dimensional case are presented, and applied to finite-dimensional linear programming problems. Sections 5 and 6 are devoted to error bounds for second order cone and semidefinite programming, respectively. Then in Section 7 we investigate conic optimization problems with block structured variables. In Section 8 verified certificates of infeasibility are presented, and in the following section we will focus on some applications in combinatorial optimization. Section 10 contains numerical results for the NETLIB Linear Programming Library (obtained by the C++ software package Lurupa [17]) and for the SDPLIB benchmark problems (obtained by the MATLAB software package VSDP [15]). Finally, some conclusions are given.

2 Notation and Preliminaries

Let \mathcal{X} be a real vector space equipped with a norm $\|\cdot\|$, and let $\mathcal{K} \subseteq \mathcal{X}$ be a *convex cone*, i.e.

$$\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}, \alpha\mathcal{K} \subseteq \mathcal{K} \text{ for } \alpha \in \mathbf{R}_+, \quad (1)$$

where \mathbf{R}_+ denotes the set of nonnegative real numbers. A convex cone \mathcal{K} induces a *partial ordering*

$$x \leq y \iff y - x \in \mathcal{K}, \quad (2)$$

which is a transitive and reflexive binary relation on \mathcal{X} compatible with addition and scalar multiplication:

$$x \leq y, u \leq v, \alpha \in \mathbf{R}_+ \implies x + u \leq y + v \text{ and } \alpha x \leq \alpha y. \quad (3)$$

Conversely, each partial ordering determines a convex cone, namely the *positive cone*

$$\mathcal{K} := \{x \in \mathcal{X} : x \geq 0\}. \quad (4)$$

A vector space \mathcal{X} equipped with a partial ordering is called a *partially ordered vector space*. A partial ordering is called *antisymmetric*, if

$$x \leq y, y \leq x \implies x = y.$$

It can be proved that antisymmetric partial orderings correspond to *pointed cones*, i.e.

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}.$$

If not explicitly mentioned we do not assume that the partial ordering is antisymmetric. Given a partial ordering the set

$$[\underline{x}, \bar{x}] := \{x \in \mathcal{X} : \underline{x} \leq x \leq \bar{x}\} = (\underline{x} + \mathcal{K}) \cap (\bar{x} - \mathcal{K}) \quad (5)$$

is called an *interval*. For a subset \mathcal{M} of a partially ordered vector space \mathcal{X} a vector \underline{x} is called a *lower bound* of \mathcal{M} , if $\underline{x} \leq m$ for all $m \in \mathcal{M}$, and in this case we write $\underline{x} \leq \mathcal{M}$. The lower bound \underline{x} is called *infimum* of \mathcal{M} if every other lower bound \underline{y} of \mathcal{M} satisfies $\underline{y} \leq \underline{x}$. Analogously, *upper bounds* and *supremum* are defined. \mathcal{X} is said to be a *vector lattice* for the partial ordering \leq if for all $x, y \in \mathcal{X}$ the supremum $\sup\{x, y\}$ and the infimum $\inf\{x, y\}$ exists and is contained in \mathcal{X} , respectively. In a vector lattice the operations $x^+ := \sup\{x, 0\}$, $x^- := \inf\{x, 0\}$ and $|x| := \sup\{x, -x\}$ are defined, and the properties $|x| = x^+ - x^-$, $x = x^+ + x^-$, $|x| = 0$ iff $x = 0$, $|\lambda x| = |\lambda| |x|$ for real λ , and $|x + y| \leq |x| + |y|$ are satisfied.

Let \mathcal{X}^* denote the *dual space* of \mathcal{X} , that is the space of continuous linear functionals endowed with the operator norm. The set \mathcal{K}^* of all positive linear functionals, i.e.

$$\mathcal{K}^* = \{y \in \mathcal{X}^* : \langle y, x \rangle := y(x) \geq 0 \text{ for all } x \in \mathcal{K}\}, \quad (6)$$

is a convex cone in \mathcal{X}^* defining a partial ordering in the dual space.

The basic properties and relations for vector lattices as well as examples can be found in Birkhoff [3] and Bourbaki [5]; see also Peressini [35] and Schaefer [43]. We use the same notation $\|\cdot\|$ and \leq for all norms and partial orderings. It will always be clear from the context which norm and which cone is referred to. Hence, if $x \in \mathcal{X}$ then $x \geq 0$ means $x \in \mathcal{K}$, and if $y \in \mathcal{X}^*$ then $y \geq 0$ denotes $y \in \mathcal{K}^*$. Observe that we do not write y^* for a continuous linear functional in \mathcal{X}^* because from the position in $\langle y, x \rangle$ the meaning is clear, and we can omit the star. This notion is closely related to Hilbert spaces and the Theorem of Riesz which states that the continuous linear functions can be represented by the inner product $\langle y, x \rangle$ where y is a vector in the Hilbert space.

In the following some illustrative and well-known examples of normed vector lattices are shown. The real finite dimensional space $\mathcal{X} = \mathbf{R}^n$ equipped with the Euclidean inner product and the Euclidean norm $\|\cdot\|$ can be ordered by the convex cone

$$\mathcal{K} := \mathbf{R}_+^n = \{x \in \mathbf{R}^n : x_i \geq 0 \text{ for } i = 1, \dots, n\}. \quad (7)$$

This cone is *self-dual* (i.e. $\mathcal{K} = \mathcal{K}^*$) and implies the lattice operations

$$x_i^+ = \max\{0, x_i\}, \quad x_i^- = \min\{0, x_i\}, \quad |x_i| = x_i^+ - x_i^- \quad (8)$$

for $i = 1, \dots, n$. This vector lattice is used in *linear programming* (LP).

In *second order cone programming* (SOCP) the same normed space $\mathcal{X} = \mathbf{R}^n$ is equipped with the partial ordering defined by the convex *ice-cream* or *Lorenz cone*

$$\mathcal{K} := \left\{ x = \begin{pmatrix} x_{\cdot} \\ x_n \end{pmatrix} \in \mathbf{R}^n : x_n \geq \|x_{\cdot}\| \right\}, \quad (9)$$

where $x_{\cdot} := (x_1, \dots, x_{n-1})^T$. This cone is also self-dual and further properties are described in Section 5.

In *semidefinite programming* (SDP) the real linear space \mathcal{X} is $\mathbf{R}^{n(n+1)/2}$, which is identified with the set of real symmetric $n \times n$ matrices X . The inner product of X, Y is defined by $\langle X, Y \rangle := \text{trace} X^T Y = \sum_{ij} X_{ij} Y_{ij}$, and the induced norm $\|X\| := (\text{trace} X^T X)^{\frac{1}{2}}$ is the *Frobenius norm*.

The space $\mathcal{X} = \mathbf{R}^{n(n+1)/2}$ is a Hilbert space, thus self-dual, and it is equipped with the self-dual cone of positive semidefinite matrices

$$\mathcal{K} := S_+^n = \{X \in \mathcal{X} : X \text{ is positive semidefinite}\}. \quad (10)$$

Using the eigenvalue decomposition $X = Q^T \Lambda Q$ of a real symmetric matrix it follows that

$$X^- = Q^T \Lambda^- Q, \quad X^+ = Q^T \Lambda^+ Q, \quad |X| = Q^T |\Lambda| Q, \quad (11)$$

where Λ^- , Λ^+ , and $|\Lambda|$ denote the diagonal matrices with nonpositive, nonnegative, and modulus of the eigenvalues of X on the diagonal, respectively.

Occasionally, it is useful to represent symmetric matrices X as column vectors x by using the *svec* operator:

$$x := \text{svec}(X) := (X_{11}; \sqrt{2}X_{21}; \dots; \sqrt{2}X_{n1}; X_{22}; \sqrt{2}X_{32}; \dots; X_{nn}). \quad (12)$$

Here we follow the convention of MATLAB and use “;” for adjoining vectors in a column. Then the inner product between symmetric matrices X and Y is the usual inner product, that is

$$\langle X, Y \rangle = x^T y. \quad (13)$$

We also use the notation $x \in \mathcal{K}$ and $x \leq y$ if the corresponding symmetric matrices X and Y such that $x = \text{svec}(X)$ and $y = \text{svec}(Y)$ have these properties.

For any compact Hausdorff space Ω , the vector space $\mathcal{X} := C(\Omega)$ of real-valued functions is a normed vector lattice with norm

$$\|f\|_{C(\Omega)} := \sup_{x \in \Omega} \{|f(x)|\}$$

and ordering cone

$$\mathcal{K} := \{f \in C(\Omega) : f(x) \geq 0 \text{ for all } x \in \Omega\}.$$

Finally, we mention $L_p(\Omega)$, the vector space of Lebesgue-integrable functions $f : \Omega \rightarrow \mathbf{R}$, where $\Omega \subseteq \mathbf{R}^n$, $1 \leq p < \infty$. This space is equipped with the norm

$$\|f\|_p := \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}},$$

and can be partially ordered by the cone

$$\mathcal{K} := \{f \in L_p(\Omega) : f(x) \geq 0 \text{ almost everywhere on } \Omega\}.$$

This yields a normed vector lattice, which is of interest in the case of Volterra and Fredholm type equations.

3 Conic Optimization Problems

We study rigorous error bounds for the *conic optimization problem in standard form*

$$\text{minimize } \langle c, x \rangle \quad \text{s.t.} \quad Ax = b, \quad x \in \mathcal{K}, \quad (14)$$

where \mathcal{X} is a real normed vector space, $\mathcal{K} \subseteq \mathcal{X}$ is a convex cone, $c \in \mathcal{X}^*$, \mathcal{Y} is a real normed vector space, A denotes a continuous linear operator from \mathcal{X} to \mathcal{Y} , and $b \in \mathcal{Y}$. With \widehat{f}_p

we denote the primal optimal value, where $\widehat{f}_p := +\infty$ if the problem is infeasible. Many interesting examples of optimization problems can be formulated in this framework. In the following some familiar facts are described.

The *Lagrangian function* of problem (14) has the form

$$L(x, y) := \langle c, x \rangle + \langle y, b - Ax \rangle, \quad (15)$$

where $y \in \mathcal{Y}^*$. The optimization problem

$$\inf_{x \in \mathcal{K}} \sup_{y \in \mathcal{Y}^*} L(x, y) \quad (16)$$

is equivalent to (14). Indeed, if $b - Ax = 0$ then $\langle y, b - Ax \rangle = 0$ for each $y \in \mathcal{Y}^*$, and the supremum of $L(x, y)$ is equal to $\langle c, x \rangle$. In the case where $b - Ax \neq 0$ there is some y with $\langle y, b - Ax \rangle > 0$, and hence the supremum is infinite.

Obviously, the Lagrangian satisfies $L(x, y) = \langle y, b \rangle + \langle -A^*y + c, x \rangle$ where A^* is the *adjoint operator*. By exchanging in (16) infimum and supremum we obtain the dual problem

$$\sup_{y \in \mathcal{Y}^*} \inf_{x \in \mathcal{K}} L(x, y) \quad (17)$$

with optimal value \widehat{f}_d . Since exchanging inf and sup always produces a lower bound, *weak duality* holds, that is $\widehat{f}_d \leq \widehat{f}_p$. Because $\inf_{x \in \mathcal{K}} L(x, y) = -\infty$ whenever $-A^*y + c \notin \mathcal{K}^*$ the dual problem can be written equivalently in the form

$$\text{maximize } \langle y, b \rangle \text{ s.t. } -A^*y + c \in \mathcal{K}^*, y \in \mathcal{Y}^*. \quad (18)$$

We set $\widehat{f}_d := -\infty$, if the dual problem is infeasible.

Let x be primal feasible, and let y be dual feasible, then

$$\langle c, x \rangle = \langle c, x \rangle + \langle y, b - Ax \rangle = \langle -A^*y + c, x \rangle + \langle y, b \rangle \geq \langle y, b \rangle, \quad (19)$$

and hence equality holds iff the *complementarity condition*

$$\langle -A^*y + c, x \rangle = 0 \quad (20)$$

are fulfilled. This condition means that the feasible pair x, y is a saddle point of the Lagrangian. Moreover, it follows that there is no duality gap between the primal and the dual problem, and both problems have optimal solutions if and only if there exists a primal and a dual feasible solution fulfilling the complementarity conditions. In other cases where such primal dual feasible pairs do not exist strong duality may be not fulfilled.

Duality theory is central to the study of optimization. First, algorithms are frequently based on duality (like primal-dual interior point methods), secondly they enable to check whether a given feasible point is optimal, and thirdly it allows to perform a sensitivity analysis. For more results on duality theory in the infinite-dimensional case see for example Renegar [36], Rockafellar [38], and the literature cited there.

4 Lower and Upper Bounds for the Optimal Value

This section is elementary but important for understanding both, the basic ideas behind rigorous forward error bounds and implementations. It turns out that for computing these error bounds only approximate primal and dual solutions \tilde{x} and \tilde{y} are required. Further assumptions about the accuracy of the approximations are not necessary; they need to be neither primal nor dual feasible. If the accuracy is poor, however, then the error bounds cause overestimation.

The cones \mathcal{K} and \mathcal{K}^* create partial orderings for the vector spaces \mathcal{X} and \mathcal{X}^* , respectively. *We assume in this paper that for subsets of these partially ordered vector spaces there exist lower and upper bounds. If the existence of the infimum or the supremum is necessary we mention this explicitly.* Note that all vector lattices satisfy this assumption.

We start with a simple result concerning bounds for linear functionals.

Lemma 4.1 *Assume that $x, \bar{x} \in \mathcal{K}$, $x \leq \bar{x}$, and let $d, \underline{d}^- \in \mathcal{X}^*$ with $\underline{d}^- \leq \{d, 0\}$. Then*

$$\langle d, x \rangle \geq \langle \underline{d}^-, \bar{x} \rangle \quad \text{and} \quad \langle \underline{d}^-, \bar{x} \rangle \leq 0. \quad (21)$$

PROOF. Since $0 \leq d - \underline{d}^-$ and $x \geq 0$ it is

$$0 \leq \langle d - \underline{d}^-, x \rangle = \langle d, x \rangle - \langle \underline{d}^-, x \rangle.$$

Hence $\langle d, x \rangle \geq \langle \underline{d}^-, x \rangle$. Since $-\underline{d}^- \geq 0$ and $\bar{x} - x \geq 0$ the linearity of the functional $-\underline{d}^-$ yields

$$0 \leq \langle -\underline{d}^-, \bar{x} - x \rangle = \langle \underline{d}^-, x \rangle - \langle \underline{d}^-, \bar{x} \rangle,$$

which immediately implies (21). □

The dual version of this lemma is:

Lemma 4.2 *Assume that $x, \underline{x}^- \in \mathcal{X}$ with $\underline{x}^- \leq \{x, 0\}$, and let $d, \bar{d} \in \mathcal{K}^*$ with $d \leq \bar{d}$. Then*

$$\langle d, x \rangle \geq \langle \bar{d}, \underline{x}^- \rangle \quad \text{and} \quad \langle \bar{d}, \underline{x}^- \rangle \leq 0. \quad (22)$$

PROOF. Since $0 \leq x - \underline{x}^-$ and $d \geq 0$ it is

$$0 \leq \langle d, x - \underline{x}^- \rangle = \langle d, x \rangle - \langle d, \underline{x}^- \rangle.$$

Hence $\langle d, x \rangle \geq \langle d, \underline{x}^- \rangle$. Since $-\underline{x}^- \geq 0$ and $\bar{d} - d \geq 0$ it follows that

$$0 \leq \langle \bar{d} - d, -\underline{x}^- \rangle = \langle d, \underline{x}^- \rangle - \langle \bar{d}, \underline{x}^- \rangle$$

which implies (22). □

Lemma 4.3 Assume that \mathcal{X} and \mathcal{X}^* are normed vector lattices. Assume that for $x, \bar{x} \in \mathcal{X}$ and $d, \bar{d} \in \mathcal{X}^*$ the inequalities

$$|x| \leq \bar{x} \quad \text{and} \quad |d| \leq \bar{d} \quad (23)$$

are satisfied. Then

$$|\langle d, x \rangle| \leq \langle \bar{d}, \bar{x} \rangle \leq \|\bar{d}\| \|\bar{x}\|.$$

PROOF. Since $|x| = \sup\{x, -x\} \leq \bar{x}$ it follows that

$$-\bar{x} \leq x \leq \bar{x}.$$

Analogously we obtain

$$-\bar{d} \leq d \leq \bar{d}.$$

The inequalities $\bar{x} - x \geq 0$ and $\bar{d} - d \geq 0$ imply

$$0 \leq \langle \bar{d} - d, \bar{x} - x \rangle = \langle \bar{d}, \bar{x} \rangle - \langle \bar{d}, x \rangle - \langle d, \bar{x} \rangle + \langle d, x \rangle,$$

and the inequalities $x + \bar{x} \geq 0$ and $d + \bar{d} \geq 0$ imply

$$0 \leq \langle d + \bar{d}, x + \bar{x} \rangle = \langle d, x \rangle + \langle d, \bar{x} \rangle + \langle \bar{d}, x \rangle + \langle \bar{d}, \bar{x} \rangle.$$

Adding both inequalities yields

$$0 \leq 2(\langle \bar{d}, \bar{x} \rangle + \langle d, x \rangle),$$

and therefore

$$-\langle d, x \rangle \leq \langle \bar{d}, \bar{x} \rangle.$$

Because of the symmetry of x and $-x$ in the definition $|x| = \sup\{x, -x\}$ it follows that

$$\langle d, x \rangle = -\langle d, -x \rangle \leq \langle \bar{d}, \bar{x} \rangle.$$

Hence

$$|\langle d, x \rangle| \leq \langle \bar{d}, \bar{x} \rangle,$$

and the last inequality follows from the definition of the norm of an operator. \square

For bounding rigorously the optimal value, we claim boundedness qualifications, which are more suitable for our purpose than Slater's constraint qualifications. We assume that the conic optimization problem satisfies the following condition which we call *primal boundedness qualification* (PBQ):

- (i) Either the primal problem is infeasible,
- (ii) or \widehat{f}_p is finite, and there is a simple bound $\bar{x} \in \mathcal{K}$ such that for every $\varepsilon > 0$ there exists a primal feasible solution $x(\varepsilon)$ satisfying $x(\varepsilon) \leq \bar{x}$ and $\langle c, x(\varepsilon) \rangle - \widehat{f}_p \leq \varepsilon$

Observe that PBQ implies that the primal problem is bounded from below, but the existence of an optimal solution is not demanded, only simple bounds \bar{x} for ε -optimal solutions are required. The following theorem provides a finite lower bound \underline{f}_p of the primal optimal value.

Theorem 4.1 *Assume that PBQ holds. Let $\tilde{y} \in \mathcal{Y}^*$ and let $d := -A^*\tilde{y} + c$. Suppose further that $\underline{d}^- \leq \{d, 0\}$, then:*

(a) *The primal optimal value is bounded from below by*

$$\hat{f}_p \geq \langle \tilde{y}, b \rangle + \langle \underline{d}^-, \bar{x} \rangle =: \underline{f}_p \quad (24)$$

(b) *If $\underline{d}^- = 0$, then \tilde{y} is dual feasible and $\hat{f}_d \geq \underline{f}_p = \langle \tilde{y}, b \rangle$, and if moreover \tilde{y} is optimal then $\hat{f}_d = \underline{f}_p$.*

PROOF. (a) If the primal problem is infeasible, then $\hat{f}_p = +\infty$, and each finite value is a lower bound. Hence, assume that PBQ (ii) is satisfied with $x := x(\varepsilon)$ and $\varepsilon > 0$. Then

$$\begin{aligned} \langle c, x \rangle &= \langle d, x \rangle + \langle A^*\tilde{y}, x \rangle \\ &= \langle \tilde{y}, b \rangle + \langle A^*\tilde{y}, x \rangle - \langle \tilde{y}, b \rangle + \langle d, x \rangle \\ &= \langle \tilde{y}, b \rangle + \langle \tilde{y}, Ax - b \rangle + \langle d, x \rangle. \end{aligned}$$

Since x is primal feasible $Ax - b = 0$, and

$$\langle c, x \rangle = \langle \tilde{y}, b \rangle + \langle d, x \rangle.$$

Lemma 4.1 implies the inequality

$$\langle c, x \rangle \geq \langle \tilde{y}, b \rangle + \langle \underline{d}^-, \bar{x} \rangle.$$

Because of PBQ (ii)

$$\hat{f}_p \geq \langle c, x \rangle - \varepsilon \geq \langle \tilde{y}, b \rangle + \langle \underline{d}^-, \bar{x} \rangle - \varepsilon.$$

For $\varepsilon \rightarrow 0$ the assertion (a) follows.

(b) If $\underline{d}^- = 0$ then $d \in \mathcal{K}^*$, implying that \tilde{y} is dual feasible, and the assertion follows. \square

In particular, an approximate solution \tilde{y} which is close to optimality implies that d is close to \mathcal{K}^* . Hence, each lower bound \underline{d}^- sufficiently close to d^- is almost zero, and it follows that $\underline{f}_p \approx \langle \tilde{y}, b \rangle$ is reasonable; that is the overestimation is not very much larger than necessary. The lower bound uses the approximate optimal value $\langle \tilde{y}, b \rangle$, and a correction is added which takes into account the violation of dual feasibility \underline{d}^- evaluated at the upper bound \bar{x} .

We illustrate the bound for linear programming problems in standard form

$$\text{minimize } c^T x \quad \text{s.t.} \quad Ax = b, x \geq 0. \quad (25)$$

This is the special case of the conic optimization problem where $\mathcal{X} = \mathcal{X}^* = \mathbf{R}^n$, $\mathcal{K} = \mathcal{K}^* = \mathbf{R}_+^n$ and $\mathcal{Y} = \mathcal{Y}^* = \mathbf{R}^m$. It is well-known that for linear programming strong duality $\widehat{f}_p = \widehat{f}_d =: \widehat{f}$ holds without any constraint qualifications. Hence, Theorem 4.1 yields immediately the lower bound

$$\widehat{f} \geq b^T \widetilde{y} + (\underline{d}^-)^T \overline{x} =: \underline{f}_p, \quad (26)$$

where $\underline{d}_j^- \leq \min\{0, (-A^T \widetilde{y} + c)_j\}$ for $j = 1, \dots, n$. It is straightforward to control all effects of rounding errors for computing \underline{f}_p by using directed rounding or interval arithmetic. The MATLAB toolbox INTLAB [42] provides the directed rounding modes, and the following short INTLAB program produces a rigorous lower bound:

```
setround(-1);
dlminus = min(0,A'*(-yt)+c);
flow = b'*yt + dlminus'*xup;
setround(0);
```

If interval arithmetic is used, then the input data A, b, c may be intervals, and we obtain a lower bound for each instance within the interval data. Verified error bounds for general linear programming problems also with free variables can be found in [14], and for formula (26) see Corollary 6.1 in [14].

To compute a rigorous upper bound of the optimal value we assume that the conic optimization problem satisfies the following condition, which we call the *dual boundedness qualification* (DBQ):

- (i) Either the dual problem is infeasible,
- (ii) or \widehat{f}_d is finite, and there is a simple bound \overline{y} such that for every $\varepsilon > 0$ there exists a dual feasible solution $y(\varepsilon)$ satisfying $|y(\varepsilon)| \leq \overline{y}$ and $\widehat{f}_d - \langle y(\varepsilon), b \rangle \leq \varepsilon$

Theorem 4.2 *Assume that DBQ holds. Let $\tilde{x} \in X$, and suppose further that*

$$|A\tilde{x} - b| \leq \overline{r}, \quad (27)$$

$$\underline{x}^- \leq \{\tilde{x}, 0\}, \text{ and} \quad (28)$$

$$\overline{d} \geq -A^*y + c \text{ for all dual feasible } y \text{ with } |y| \leq \overline{y}. \quad (29)$$

Then:

(a) The dual optimal value is bounded from above by

$$\widehat{f}_d \leq \langle c, \tilde{x} \rangle - \langle \bar{d}, \underline{x}^- \rangle + \langle \bar{y}, \bar{r} \rangle =: \bar{f}_d. \quad (30)$$

(b) If $\underline{x}^- = 0$ and $\bar{r} = 0$, then \tilde{x} is primal feasible and $\widehat{f}_p \leq \bar{f}_d = \langle c, \tilde{x} \rangle$, and if moreover \tilde{x} is optimal, then $\widehat{f}_p = \bar{f}_d$.

PROOF. (a) If the dual problem is infeasible then $\widehat{f}_d = -\infty$, and each finite value is an upper bound. Hence, assume that DBQ (ii) is satisfied with $y := y(\varepsilon)$ and $\varepsilon > 0$. Then

$$\begin{aligned} \langle y, b \rangle &= -\langle y, A\tilde{x} - b \rangle + \langle y, A\tilde{x} \rangle \\ &= \langle c, \tilde{x} \rangle + \langle y, A\tilde{x} \rangle - \langle c, \tilde{x} \rangle - \langle y, A\tilde{x} - b \rangle \\ &= \langle c, \tilde{x} \rangle - \langle c - A^*y, \tilde{x} \rangle - \langle y, A\tilde{x} - b \rangle. \end{aligned}$$

Since y is dual feasible and $\bar{d} \geq d := -A^*y + c \geq 0$, we can apply Lemmata 4.2 and 4.3 which yield

$$\langle y, b \rangle \leq \langle c, \tilde{x} \rangle - \langle \bar{d}, \underline{x}^- \rangle + \langle \bar{y}, \bar{r} \rangle.$$

Because of DBQ (ii)

$$\widehat{f}_d \leq \langle y, b \rangle + \varepsilon \leq \langle c, \tilde{x} \rangle - \langle \bar{d}, \underline{x}^- \rangle + \langle \bar{y}, \bar{r} \rangle + \varepsilon.$$

For $\varepsilon \rightarrow 0$ we obtain the upper bound (30).

The assertion (b) follows immediately, since $\underline{x}^- = 0$ and $\bar{r} = 0$. \square

Observe that for finite dimensional \mathcal{Y} or in the case where \mathcal{Y} is a vector lattice the absolute value $|\cdot|$ is defined. If the absolute value is not available, then we replace the inequalities $|y(\varepsilon)| \leq \bar{y}$, $|A\tilde{x} - b| \leq \bar{r}$ by $\|y(\varepsilon)\| \leq \bar{y}$ and $\|A\tilde{x} - b\| \leq \bar{r}$, respectively, and we obtain the error bound

$$\widehat{f}_d \leq \langle c, \tilde{x} \rangle - \langle \bar{d}, \underline{x}^- \rangle + \bar{y} \cdot \bar{r} =: \bar{f}_d. \quad (31)$$

Similarly as in the case of the lower bound, the upper bound uses the approximate value $\langle c, \tilde{x} \rangle$ and takes into account the violations of \tilde{x} wrt. to the cone \mathcal{K} and the linear equations.

The computation of the quantity $\langle \bar{d}, \underline{x}^- \rangle$ can be avoided. Since \tilde{x} is an approximate optimal solution, $\tilde{x} \in \mathcal{K}$ or \tilde{x} is close to \mathcal{K} (provided the conic solver has computed reasonable approximations). Hence, for the supremum $\tilde{x}^+ = \sup\{\tilde{x}, 0\}$ the distance $\|\tilde{x} - \tilde{x}^+\|$ is small. If we replace in Theorem 4.2 \tilde{x} by \tilde{x}^+ then the quantity $|\langle c, \tilde{x}^+ \rangle - \langle c, \tilde{x} \rangle|$ is small, but $\underline{x}^- := 0 \leq \{\tilde{x}^+, 0\}$ yielding $\langle \bar{d}, \underline{x}^- \rangle = 0$ and the upper bound

$$\widehat{f}_d \leq \langle c, \tilde{x}^+ \rangle + \langle \bar{y}, \bar{r} \rangle, \quad (32)$$

where $|A\tilde{x}^+ - b| \leq \bar{r}$. In general it is not possible to compute the supremum \tilde{x}^+ exactly, but each close upper bound $\tilde{\tilde{x}} \geq \tilde{x}^+$ will suffice.

In the special case of linear programming we can take the exact supremum $\tilde{x}^+ \in \mathbf{R}_+^n$ defined by (8). Then $\underline{x}^- = 0$, and we obtain the upper bound

$$\hat{f} \leq c^T \tilde{x}^+ + \bar{y}^T \bar{r} = \bar{f}_d. \quad (33)$$

The following short INTLAB program produces this upper bound:

```
xtplus = max(0,xt)
setround(-1);
rn = abs(A*xtplus -b);
setround(+1);
rp = abs(A*xtplus -b);
r = max(rn,rp);
fu = c'*xtplus + yup'*r;
setround(0);
```

Until now we have assumed the existence of ε -optimal solutions within some reasonable bounds. Now we mention briefly that in the case where appropriate primal or dual boundedness qualifications are not known it is frequently possible to compute verified primal and dual feasible solutions which are close to optimality. These solutions can be used to compute verified reasonable error bounds for the optimal value. The basic algorithm consists of the following steps:

- (i) Perturb the original problem slightly such that the optimal solution of the perturbed problem is an interior feasible solution of the original problem.
- (ii) Solve the perturbed problem approximately.
- (iii) Use this approximation to compute an enclosure (i.e an appropriate interval) containing a feasible solution.
- (iv) Evaluate the objective function for the enclosure.

Especially step (iii) is nontrivial since the existence of feasible solutions must be rigorously proved. Interval arithmetic provides several methods for computing enclosures of solutions for linear and nonlinear systems in the finite dimensional case, but also for infinite dimensional problems (certain types of ordinary and partial differential equations) enclosure methods are known. However the bounds obtained in this way have two disadvantages. They are much more time-consuming than the previous ones, and they provide an upper bound of the primal optimal value only if the primal problem is well-posed, and a lower bound of the dual optimal value only if the dual problem is well-posed.

A detailed description of this algorithm can be found in the case of linear programming in [14], for smooth convex programming problems in [13], and for semidefinite programming problems and linear matrix inequalities in [16].

5 Second Order Cone Programming

In SOCP the partial ordering is defined by the ice-cream cone (9) yielding a finite dimensional vector lattice equipped with the following operations.

Theorem 5.1 *Let $\mathcal{K} \subseteq \mathbf{R}^n$ be defined by (9). Then for $x \in \mathbf{R}^n$*

$$x^+ = \begin{cases} x & \text{if } x_n \geq \|x_{\cdot}\| \\ 0 & \text{if } x_n \leq -\|x_{\cdot}\| \\ \frac{\|x_{\cdot}\| + x_n}{2\|x_{\cdot}\|} \begin{pmatrix} x_{\cdot} \\ \|x_{\cdot}\| \end{pmatrix} & \text{if } -\|x_{\cdot}\| < x_n < \|x_{\cdot}\| \end{cases} \quad (34)$$

and

$$x^- = \begin{cases} x & \text{if } x_n \leq -\|x_{\cdot}\| \\ 0 & \text{if } x_n \geq \|x_{\cdot}\| \\ -\frac{\|x_{\cdot}\| - x_n}{2\|x_{\cdot}\|} \begin{pmatrix} -x_{\cdot} \\ \|x_{\cdot}\| \end{pmatrix} & \text{if } -\|x_{\cdot}\| < x_n < \|x_{\cdot}\| \end{cases} \quad (35)$$

PROOF. First we prove (34). If $x_n \geq \|x_{\cdot}\|$ then $x \in \mathcal{K}$ which implies $x^+ := \sup\{x, 0\} = x$. If $x_n \leq -\|x_{\cdot}\|$ then $x \in -\mathcal{K}$ and $x^+ = 0$. Finally, assume that $-\|x_{\cdot}\| < x_n < \|x_{\cdot}\|$. Then $x_{\cdot} \neq 0$, and a simple geometric argument shows that x^+ is the orthogonal projection of x onto the boundary of \mathcal{K} , that is

$$x^+ = \alpha \begin{pmatrix} x_{\cdot} \\ \|x_{\cdot}\| \end{pmatrix} \quad \text{and} \quad 0 = (x^+ - x)^T x^+.$$

The latter condition describes the orthogonality. Since

$$\begin{aligned} 0 &= \begin{pmatrix} \alpha x_{\cdot} & -x_{\cdot} \\ \alpha \|x_{\cdot}\| & -x_n \end{pmatrix}^T \begin{pmatrix} \alpha x_{\cdot} \\ \alpha \|x_{\cdot}\| \end{pmatrix} \\ &= \alpha^2 \|x_{\cdot}\|^2 - \alpha \|x_{\cdot}\|^2 + \alpha^2 \|x_{\cdot}\|^2 - \alpha x_n \|x_{\cdot}\| \\ &= 2\alpha^2 \|x_{\cdot}\|^2 - \alpha(\|x_{\cdot}\|^2 + x_n \|x_{\cdot}\|), \end{aligned}$$

we get $\alpha = (\|x_\cdot\| + x_n)/2\|x_\cdot\|$ which proves (34).

Finally, (35) follows from (34) since $x^- = \inf\{x, 0\} = -\sup\{-x, 0\}$. \square

Due to rounding-off errors x^- may not be computed exactly. But for computing rigorous results we know from the previous section that it is sufficient to compute a lower bound $\underline{x}^- \leq x^-$ by using directed rounding or interval arithmetic. The distinction of cases, however, must be implemented carefully when x_n is almost equal to $-\|x_\cdot\|$ or $\|x_\cdot\|$. A similar remark applies to the computation of an upper bound $\bar{x}^+ \geq x^+$.

Let \mathcal{K} be the Cartesian product of ice-cream cones $\mathcal{K}_j \subseteq \mathbf{R}^{n_j}$ for $j = 1, \dots, n$. This is a convex, self-dual cone (see Alizadeh, Goldfarb [2]). The *standard SOCP problem* has the form

$$\text{minimize } \sum_{j=1}^n c_j^T x_j \quad \text{s.t.} \quad \sum_{j=1}^n A_j x_j = b, \quad x_j \in \mathcal{K}_j \quad \text{for } j = 1, \dots, n, \quad (36)$$

where $A_j \in \mathbf{R}^{m \times n_j}$, $c_j, x_j \in \mathbf{R}^{n_j}$ and $b \in \mathbf{R}^m$. If we merge these quantities

$$\begin{aligned} A &:= (A_1; \dots; A_n), \\ c &:= (c_1; \dots; c_n), \\ x &:= (x_1; \dots; x_n), \end{aligned} \quad (37)$$

then the standard SOCP problem has the form (14), and it follows that the dual problem (18) can be written as

$$\text{minimize } b^T y \quad \text{s.t.} \quad -A_j^T y + c_j \in \mathcal{K}_j \quad \text{for } j = 1, \dots, n. \quad (38)$$

Here we have chosen the finite-dimensional spaces $\mathcal{X} := \mathbf{R}^{\bar{n}}$ where $\bar{n} = \sum_j n_j$ and $\mathcal{Y} := \mathbf{R}^m$ equipped with the Euclidean inner products. By $x_{i,j}$ we denote the i -th component of the vector x_j , and $x_{:,j} := (x_{1,j}, \dots, x_{n_j-1,j})^T$. In this section

$$\bar{x} = (\bar{x}_1; \dots; \bar{x}_n)$$

denotes a vector in \mathcal{K} with $\bar{x}_{:,j} = 0$ and $\bar{x}_{n_j,j} > 0$ for every j . Then

$$x \leq \bar{x} \quad \Leftrightarrow \quad \|x_{:,j}\| + x_{n_j,j} \leq \bar{x}_{n_j,j} \quad \text{for } j = 1, \dots, n. \quad (39)$$

The computation of a rigorous lower bound for the optimal value of (36) is a straightforward application of Theorem 4.1.

Corollary 5.1 *Assume that PBQ holds for some $\bar{x} \in \mathcal{K}$. Let $\tilde{y} \in \mathbf{R}^m$, and let*

$$d_j := -A_j^T \tilde{y} + c_j \quad \text{for } j = 1, \dots, n. \quad (40)$$

Suppose further that for $j = 1, \dots, n$ there are lower bounds $\underline{d}_j^- \leq d_j^-$. Then:

(a) The primal optimal value is bounded from below by

$$\hat{f}_p \geq b^T \tilde{y} + \sum_{j=1}^n \underline{d}_{n_j,j}^- \bar{x}_{n_j,j} := \underline{f}_p. \quad (41)$$

(b) If $d_{n_j,j} \geq \|d_{:,j}\|$ for $j = 1, \dots, n$, then \tilde{y} is dual feasible and $\hat{f}_d \geq \underline{f}_p = b^T \tilde{y}$, and if moreover \tilde{y} is optimal then $\hat{f}_d = \underline{f}_p$.

PROOF.

It follows from (24) that

$$\begin{aligned} \hat{f}_p &\geq \langle \tilde{y}, b \rangle + \langle \underline{d}^-, \bar{x} \rangle \\ &= b^T \tilde{y} + \sum_{j=1}^n \langle \underline{d}_j^-, \bar{x}_j \rangle \\ &= b^T \tilde{y} + \sum_{j=1}^n \underline{d}_{n_j,j}^- \bar{x}_{n_j,j}, \end{aligned}$$

where the last equation is fulfilled because $\bar{x}_{:,j} = 0$. This finishes the proof of (a). If $d_{n_j,j} \geq \|d_{:,j}\|$ for $j = 1, \dots, n$ then $-A_j^T \tilde{y} + c_j \in \mathcal{K}_j$, $\underline{d}_j^- = 0$, and \tilde{y} is dual feasible. Hence, (b) is proved. \square

An upper bound for the optimal value of (36) is an immediate application of Theorem 4.2.

Corollary 5.2 Assume that DBQ is fulfilled. Let $\tilde{x} \in \mathcal{K}$, and suppose further that

$$\left| \sum_{j=1}^n A_j \tilde{x}_j - b \right| \leq \bar{r},$$

then:

(a) The dual optimal value is bounded from above by

$$\hat{f}_d \leq \sum_{j=1}^n c_j \tilde{x}_j + \bar{y}^T \bar{r} =: \bar{f}_d.$$

(b) If $\bar{r} = 0$ then \tilde{x} is primal feasible and $\hat{f}_p \leq \bar{f}_d = \sum_{j=1}^n c_j \tilde{x}_j$, and if moreover \tilde{x} is optimal then $\hat{f}_p = \bar{f}_d$.

PROOF. Since $\tilde{x} \in \mathcal{K}$, it follows that $\underline{x}^- = 0 \leq \{\tilde{x}, 0\}$. Therefore the assertion is an immediate consequence of Theorem 4.2. \square

SOCP solvers may compute an approximation \tilde{x} which is not contained in \mathcal{K} , i.e. for some j the part \tilde{x}_j is not contained in the convex cone \mathcal{K}_j . Then \tilde{x} is replaced by an upper bound of \tilde{x}^+ . As aforementioned, in floating point arithmetic formula (34) must be carefully evaluated using directed rounding such that the computed result is guaranteed to be in \mathcal{K} .

6 Semidefinite Programming

We examine the *standard primal semidefinite programming problem*

$$\text{minimize } \langle C, X \rangle \quad \text{s.t. } \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m, \quad X \in \mathcal{K}, \quad (42)$$

where C, X and A_i are real symmetric $s \times s$ matrices, $b \in \mathbf{R}^m$, $\mathcal{K} = S_+^s$, and $\langle \cdot, \cdot \rangle$ denotes the inner product (13) in the linear space of symmetric matrices. Using the svec operator (12) such that

$$c = \text{svec}(C), \quad x = \text{svec}(X), \quad a_i = \text{svec}(A_i), \quad (43)$$

we can write problem (42) equivalently in the form

$$\text{minimize } c^T x \quad \text{s.t. } Ax = b, \quad x \in \mathcal{K}, \quad (44)$$

where A is the matrix with rows a_i^T . Problem (44) has the form (14), and therefore the dual problem (18) is

$$\text{maximize } b^T y \quad \text{s.t. } -A^T y + c \in \mathcal{K}, \quad y \in \mathbf{R}^m. \quad (45)$$

The equivalent matrix notation is

$$\text{maximize } b^T y \quad \text{s.t. } -\sum_{i=1}^m y_i A_i + C \in \mathcal{K}. \quad (46)$$

Corollary 6.1 *Assume that the SDP satisfies:*

- (i) *Either the primal problem is infeasible, or*
- (ii) *there exists a nonnegative number \bar{x} such that for every $\varepsilon > 0$ there exists a primal feasible solution $X(\varepsilon) \leq \bar{x} \cdot I$ and $\langle C, X(\varepsilon) \rangle - \hat{f}_p \leq \varepsilon$,*

where I denotes the identity matrix. Let $\tilde{y} \in \mathbf{R}^m$, and let

$$D = C - \sum_{i=1}^m \tilde{y}_i A_i. \quad (47)$$

Suppose further that $\underline{d}^- \leq \{\lambda_{\min}(D), 0\}$, and that D has at most l negative eigenvalues. Then:

- (a) *The primal optimal value is bounded from below by*

$$\hat{f}_p \geq b^T \tilde{y} + l \cdot \underline{d}^- \cdot \bar{x} =: \underline{f}_p. \quad (48)$$

- (b) *If $\underline{d}^- = 0$ then \tilde{y} is dual feasible and $\hat{f}_d \geq \underline{f}_p = b^T \tilde{y}$, and if moreover \tilde{y} is optimal then $\hat{f}_d = \underline{f}_p$.*

PROOF. Let D have the eigenvalue decomposition $D = Q^T \Lambda Q$, then (11) implies $D^- = Q^T \Lambda^- Q$. Hence $\underline{D}^- := \underline{d}^- \cdot Q^T \text{sign}(-\Lambda^-) Q \leq \{D, 0\}$, where sign is $+1, 0$ or -1 if the corresponding coefficient of the matrix is positive, zero or negative, respectively. Moreover, PBQ implies $\overline{X} := \overline{x} \cdot I \geq X(\varepsilon)$. Now from Theorem 4.1 (a) it follows that

$$\widehat{f}_p \geq \langle \widetilde{y}, b \rangle + \langle \underline{D}^-, \overline{X} \rangle = b^T \widetilde{y} + l \cdot \underline{d}^- \cdot \overline{x}$$

which proves (a). The part (b) is an immediate consequence of Theorem 4.1 (b). \square

In order to control all rounding errors and to compute a verified lower bound \underline{f}_p it is necessary to compute a rigorous lower bound of the smallest eigenvalue for a symmetric matrix. Interesting references for computing rigorous bounds of some or all eigenvalues are Floudas [6], Mayer [21], Neumaier [29], and Rump [39, 40]. In VSDP we have computed the quantities l and \underline{d}^- by using Weyl's Perturbation Theorem for symmetric matrices: For an approximate eigenvalue decomposition $\widetilde{D} = \widetilde{Q}^T \widetilde{\Lambda} \widetilde{Q}$ of D with eigenvalues $\widetilde{\lambda}_i$ on the diagonal of $\widetilde{\Lambda}$, we use directed rounding or interval arithmetic for computing an error matrix $E \geq |D - \widetilde{D}|$. Then the Theorem of Weyl implies that

$$|\lambda_i(D) - \widetilde{\lambda}_i| \leq \|E\|_2$$

for each eigenvalue $\lambda_i(D)$. Therefore, we obtain the bounds

$$\widetilde{\lambda}_i - \|E\|_\infty \leq \lambda_i(D) \leq \widetilde{\lambda}_i + \|E\|_\infty \quad (49)$$

for all eigenvalues, yielding immediately the quantities \underline{d}^- and l . The short INTLAB program

```
[Qt,Lt] = eig(full(mid(D)));
E = D - Qt * intval(Lt) * Qt';
r = abss(norm(E,inf));
lambda = midrad(diag(Lt),r);
```

implements these bounds for the eigenvalues of a symmetric interval matrix D , where `eig`, `abss` and `midrad` denote the MATLAB and INTLAB routines for computing approximate eigenvalues, the absolute value, and the midpoint radius description of interval quantities, respectively.

Corollary 6.2 *Assume that DBQ is fulfilled. Let $\widetilde{X} \in \mathcal{K} = S_+^s$ and suppose further that*

$$|\langle A_i, \widetilde{X} \rangle - b_i| \leq \bar{r}_i \quad \text{for } i = 1, \dots, m. \quad (50)$$

Then:

(a) The dual optimal value is bounded from above by

$$\widehat{f}_d \leq \langle C, \widetilde{X} \rangle + \widetilde{y}^T \widetilde{r} =: \overline{f}_d. \quad (51)$$

(b) If $\widetilde{r} = 0$ then \widetilde{X} is primal feasible and $\widehat{f}_p \leq \overline{f}_d = \langle C, \widetilde{X} \rangle$, and if moreover \widetilde{X} is optimal then $\widehat{f}_p = \overline{f}_d$.

PROOF. Because $\widetilde{X}^- = 0$, this corollary is an immediate consequence of Theorem 4.2. \square

In general, SDP-solvers, compute an approximation \widetilde{X} which is not a positive semidefinite matrix, but there is a cluster of negative eigenvalues of \widetilde{X} nearby zero. In order to enforce a positive semidefinite approximation, we compute a rigorous bound $\underline{x} \leq \{\lambda_{\min}(\widetilde{X}), 0\}$. Then it follows that $\widetilde{X} - \underline{x}I \in \mathcal{K}$, i.e. is positive semidefinite, and we can use this shifted matrix in the previous corollary. Then with directed rounding it is straightforward to compute \overline{f}_d .

7 Block Structured Variables

Frequently conic optimization problems have block structured variables, that is the variables are in the Cartesian product of different cones. More precisely, there are n real normed vector spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, convex cones $\mathcal{K}_1 \subseteq \mathcal{X}_1, \dots, \mathcal{K}_n \subseteq \mathcal{X}_n$, a real normed vector space \mathcal{Y} , and n continuous linear operators $A_j : \mathcal{X}_j \rightarrow \mathcal{Y}$. Let \mathcal{X} and \mathcal{K} denote the Cartesian products of the spaces \mathcal{X}_j and the cones \mathcal{K}_j , respectively. The vectors x and c and the linear operator A are partitioned appropriately:

$$\begin{aligned} x &= (x_1; \dots; x_n) \quad \text{where } x_j \in \mathcal{X}_j, \\ c &= (c_1; \dots; c_n) \quad \text{where } c_j \in \mathcal{X}_j^*, \\ A &= (A_1; \dots; A_n). \end{aligned}$$

Defining

$$Ax := \sum_{j=1}^n A_j x_j \quad \text{and} \quad \langle c, x \rangle := \sum_{j=1}^n \langle c_j, x_j \rangle, \quad (52)$$

it follows that $A : \mathcal{X} \rightarrow \mathcal{Y}$ is a continuous linear operator, and $c \in \mathcal{X}^*$. The *primal conic optimization problem with block structured variables* has the form

$$\text{minimize } \sum_{j=1}^n \langle c_j, x_j \rangle \quad \text{s.t.} \quad \sum_{j=1}^n A_j x_j = b, \quad x_j \in \mathcal{K}_j \quad \text{for } j = 1, \dots, n, \quad (53)$$

and the dual problem is

$$\text{minimize } \langle y, b \rangle \quad \text{s.t.} \quad (-A_1^* y; \dots; -A_n^* y) + (c_1; \dots; c_n) \in \mathcal{K}_1^* \times \dots \times \mathcal{K}_n^*, \quad y \in \mathcal{Y}^*. \quad (54)$$

The most important examples are *semidefinite-quadratic-linear programs*. These are block structured problems where $\mathcal{Y} = \mathbf{R}^m$,

$$x = (x_1^s; \dots; x_{n_s}^s; x_1^q; \dots; x_{n_q}^q; x^l) \in \mathcal{X}, \quad (55)$$

and

$x_1^s, \dots, x_{n_s}^s$ are symmetric matrices of various sizes,

$x_1^q, \dots, x_{n_q}^q$ are vectors of various sizes, and

x^l is a vector.

Using (52), (53), (36) and (42) we obtain the primal problem

$$\begin{aligned} & \text{minimize} \quad \sum_{j=1}^{n_s} \langle c_j^s, x_j^s \rangle + \sum_{k=1}^{n_q} (c_k^q)^T x_k^q + (c^l)^T x^l \\ & \text{s.t.} \quad \sum_{j=1}^{n_s} \langle A_{ij}^s, x_j^s \rangle + \sum_{k=1}^{n_q} (A_{ik}^q)^T x_k^q + (A_i^l)^T x^l = b_i, \quad i = 1, \dots, m \\ & \quad \quad x_j^s \in \mathcal{K}_j^s, \quad x_k^q \in \mathcal{K}_k^q, \quad x^l \in \mathcal{K}^l \quad \forall j, k, \end{aligned} \quad (56)$$

where the matrices and vectors have appropriate dimensions, and \mathcal{K}_j^s , \mathcal{K}_k^q and \mathcal{K}^l are the convex cones of positive semidefinite matrices, the ice-cream cones, and the positive orthant, respectively.

The dual problem has the form

$$\begin{aligned} & \text{maximize} \quad \sum_{i=1}^m b_i y_i \\ & \text{s.t.} \quad - \sum_{i=1}^m A_{ij}^s y_i + c_j^s \in \mathcal{K}_j^s \quad \text{for } j = 1, \dots, n^s \\ & \quad \quad - \sum_{i=1}^m A_{ik}^q y_i + c_k^q \in \mathcal{K}_k^q \quad \text{for } k = 1, \dots, n^q \\ & \quad \quad - \sum_{i=1}^m A_i^l y_i + c^l \in \mathcal{K}^l. \end{aligned} \quad (57)$$

Observe that the set of primal feasible solutions is the Cartesian product of semidefinite, quadratic and nonnegative orthant cones intersected with an affine subspace. It is possible that n^s , n^q or the length of x^l is zero, which means that one or more of the three parts of the problem is absent.

The following two corollaries provide finite lower and upper bounds of the optimal value for block structured problems.

Corollary 7.1 *Assume that PBQ holds for some $\bar{x} = (\bar{x}_1; \dots; \bar{x}_n) \in \mathcal{K}$. Let $\tilde{y} \in \mathcal{Y}$, and assume that for $j = 1, \dots, n$*

$$d_j := -A_j^* \tilde{y} + c_j \quad (58)$$

and $\underline{d}_j^- \leq \{d_j, 0\}$. Then:

(a) The primal optimal value is bounded from below by

$$\hat{f}_p \geq \langle \tilde{y}, b \rangle + \sum_{j=1}^n \langle \underline{d}_j^-, \bar{x}_j \rangle =: \underline{f}_p. \quad (59)$$

(b) If $\underline{d}_j^- = 0$ for $j = 1, \dots, n$, then \tilde{y} is dual feasible and $\hat{f}_d \geq \underline{f}_p = \langle \tilde{y}, b \rangle$, and if moreover \tilde{y} is optimal then $\hat{f}_d = \underline{f}_p$.

PROOF. This corollary follows immediately from Theorem 4.1 by observing the linearity of the block structured variables. \square

Corollary 7.2 Assume that DBQ holds for some $\bar{y} \in \mathcal{Y}^*$. Let $\tilde{x} = (\tilde{x}_1; \dots; \tilde{x}_n) \in \mathcal{K}$ and suppose further that

$$\left| \sum_{j=1}^n A_j \tilde{x}_j - b \right| \leq \bar{r}, \quad (60)$$

then

(a) The dual optimal value is bounded from above by

$$\hat{f}_d \leq \sum_{j=1}^n \langle c_j, \tilde{x}_j \rangle + \langle \bar{y}, \bar{r} \rangle =: \bar{f}_d. \quad (61)$$

(b) If $\bar{r} = 0$ then \tilde{x} is primal feasible and $\hat{f}_p \leq \bar{f}_d$, and if moreover \tilde{x} is optimal then $\hat{f}_p = \bar{f}_d$.

PROOF. The corollary is an immediate consequence of Theorem 4.2. \square

Lower and upper bounds for the optimal value of semidefinite-quadratic-linear programs can be immediately obtained by inserting the preceding formulas for LP, SOCP and SDP.

8 Certificates of Infeasibility

A theorem of alternatives states that for two systems of equations or inequalities, one or the other system has a solution, but not both. A solution of one of the systems is called a *certificate of infeasibility* for the other which has no solution, since in principle this allows an easy check to prove infeasibility. Certificates of infeasibility are frequently computed by optimization algorithms if no feasible solutions of the primal or dual constraints exist.

Especially in the presence of equality constraints, certificates cannot be represented exactly in floating point arithmetic, and therefore approximate certificates can satisfy the constraints only within certain tolerances. This effect is amplified by rounding-off errors during the calculations for computing the approximate certificate. However, it turns out that in order to prove infeasibility by using floating-point arithmetic it is sufficient if an interval of small diameter can be given which guarantees to contain a certificate. We call such an interval a *rigorous* or *verified certificate of infeasibility*, and describe briefly how such certificates can be obtained for conic problems. We begin with two well known propositions and include the short proofs.

Proposition 8.1 *Suppose that $\tilde{y} \in Y^*$ satisfies*

$$A^*\tilde{y} \in \mathcal{K}^*, \quad \langle \tilde{y}, b \rangle < 0, \quad (62)$$

then the system of primal constraints

$$Ax = b, \quad x \in \mathcal{K} \quad (63)$$

has no solution.

PROOF. If the system (63) has a solution x , then $0 \leq \langle A^*\tilde{y}, x \rangle = \langle \tilde{y}, Ax \rangle = \langle \tilde{y}, b \rangle$ contradicting our assumption $\langle \tilde{y}, b \rangle < 0$. \square

The linear functional \tilde{y} is called a *certificate of primal infeasibility*, and represents a dual unbounded ray.

Proposition 8.2 *Suppose that $\tilde{x} \in X$ satisfies*

$$A\tilde{x} = 0, \quad \tilde{x} \in \mathcal{K}, \quad \langle c, \tilde{x} \rangle < 0 \quad (64)$$

then the system of dual constraints

$$-A^*y + c \in \mathcal{K}^*, \quad y \in Y^* \quad (65)$$

has no solution.

PROOF. If the system (65) has a solution $y \in Y^*$, then $0 \leq \langle -A^*y + c, \tilde{x} \rangle = -\langle y, A\tilde{x} \rangle + \langle c, \tilde{x} \rangle = \langle c, \tilde{x} \rangle < 0$ contradicting our assumption. Hence, system (65) has no solution. \square

The vector \tilde{x} is called a *certificate of dual infeasibility* and represents a primal unbounded ray.

Many conic solvers expose infeasibility by computing approximate unbounded rays. Given an approximate primal unbounded ray $\tilde{x} \in \mathcal{X}$, dual infeasibility is proved if the equation

and sign conditions (64) can be checked rigorously on a computer. The underdetermined equation $A\tilde{x} = 0$ is in general not exactly satisfied for floating-point certificates \tilde{x} . To obtain a rigorous certificate we proceed as follows: Let β be an approximation of $\langle c, \tilde{x} \rangle$, and assume that $\beta < 0$. Otherwise, for nonnegative β the sign condition in (64) would be even not satisfied for the approximate primal unbounded ray \tilde{x} , and this indicates that \tilde{x} is not suitable. Then we compute an interval of small diameter \mathbf{x} , also called *enclosure*, for a solution of the underdetermined linear system

$$Ax = 0 \quad \text{and} \quad \langle c, x \rangle = \beta < 0, \quad (66)$$

which is close to \tilde{x} . If $\mathbf{x} \subseteq \mathcal{K}$, then there exists an $\tilde{\tilde{x}} \in \mathbf{x}$ which fulfills the condition (64) yielding a rigorous certificate of dual infeasibility. This check depends on the special problem and requires further information about the operator A and the cone \mathcal{K} . In the three cases LP, SOCP, and SDP for the finite-dimensional linear system (66) methods of interval arithmetic can be used for computing an appropriate enclosure \mathbf{x} . For a detailed description of such an algorithm see [14]. The condition $\mathbf{x} = [\underline{x}, \bar{x}] \subseteq \mathcal{K}$ can be verified for LP by checking the equivalent condition

$$\underline{x} \geq 0, \quad (67)$$

for SOCP we check the equivalent condition

$$\underline{x}_n \geq \|\underline{x}\|, \quad (68)$$

and for SDP we check

$$\lambda_{\min}(\underline{X}) \geq 0, \quad (69)$$

where $\underline{x} = \text{svec}(\underline{X})$.

In the case of an approximate dual improving ray \tilde{y} , primal infeasibility can be rigorously proved on a computer if the condition (62) can be verified; that is, if the sign conditions

$$\langle \tilde{y}, b \rangle < 0 \quad \text{and} \quad \langle A^* \tilde{y}, x \rangle \geq 0 \quad \text{for all } x \in \mathcal{K} \quad (70)$$

can be checked reliably. As before, this check depends on the special problem and requires further information about the operator A^* and the cone \mathcal{K}^* .

It follows immediately that for LP (70) is equivalent to

$$b^T \tilde{y} < 0 \quad \text{and} \quad A^T \tilde{y} \geq 0, \quad (71)$$

for SOCP we obtain the equivalent condition

$$b^T \tilde{y} < 0 \quad \text{and} \quad (A_j^T \tilde{y})_{n_j} \geq \|(A_j^T \tilde{y})\| \quad \text{for } j = 1, \dots, n, \quad (72)$$

and for SDP we get

$$b^T \tilde{y} < 0 \quad \text{and} \quad \lambda_{\min} \left(\sum_{i=1}^m \tilde{y}_i A_i \right) \geq 0. \quad (73)$$

All three conditions can be checked rigorously by using directed rounding or interval arithmetic. The vector \tilde{y} provides a rigorous certificate which can be viewed as a degenerate interval with zero diameter.

9 Combinatorial Optimization

Linear and semidefinite programs play a very useful role in global and combinatorial optimization (Wolkowicz [47]). Several methods (for example lift-and-project methods) are known for constructing linear or semidefinite relaxations, which are used in branch-bound-and-cut algorithms to eliminate regions which do not contain global minimizers. Neumaier and Shcherbina [31] have pointed out that backward error analysis has no relevance for combinatorial programs, since slightly perturbed coefficients no longer produce problems of the same class. There, one can also find an innocent-looking linear integer problem for which the commercial high quality solver CPLEX [12] and several other state-of-the-art solvers failed. The reason is that the relaxations are not solved with sufficient accuracy and global minimizers are truncated. Hence, in order to obtain safe results, it is important to have reliable, good and cheaply computable lower bounds of the optimal value for relaxations.

Various problems like Max-Cut, Partitioning, Coloring and many others can be formulated as linear integer problems where the vector of decision variables $x \in \{-1, 1\}^n$. Tight semidefinite relaxations are obtained by lifting the vector x into the space of semidefinite matrices by the operation

$$X = xx^T. \quad (74)$$

It follows immediately that

$$X \succeq 0, \text{ diag}(X) = e, \text{ and } \text{rank}(X) = 1, \quad (75)$$

where e is the vector of ones. Dropping the condition $\text{rank}(X) = 1$ we obtain a semidefinite relaxation. Laurent and Poljak [20] have shown that for this type of relaxations $-1 \leq X_{ij} \leq 1$, and if $X_{ij} \in \{-1, 1\}$ then $X = xx^T$ where $x \in \{-1, 1\}^n$. This property establishes the tightness of these relaxations. Moreover, it follows that the primal boundedness qualification is fulfilled in the way that an optimal solution exists with $\lambda_{\max}(X) \leq n$, and thus rigorous lower bounds for the optimal value can be computed.

Sometimes, these tight relaxations are in addition ill-posed. As an example we consider Graph Partitioning Problems. These are known to be NP-hard, and finding an optimal solution is difficult. Graph Partitioning has many applications among those is VLSI design. Here, we investigate semidefinite relaxations for the special case of Equicut Problems, which have turned out to deliver tight lower bounds (see also Gruber and Rendl [11]). The general case of Graph Partitioning Problems can be treated similarly.

Given an edge-weighted graph G with an even number of vertices, the problem is to find a partitioning of the vertices into two sets of equal cardinality which minimizes the weight of the edges joining the two sets. The algebraic formulation is obtained by representing the partitioning as an integer vector $x \in \{-1, 1\}^n$ satisfying the parity condition $\sum_i x_i = 0$. Then the Equicut Problem is equivalent to

$$\min \sum_{i < j} a_{ij} \frac{1 - x_i x_j}{2} \quad \text{subject to} \quad x \in \{-1, 1\}^n, \quad \sum_{i=1}^n x_i = 0,$$

where $A = (a_{ij})$ is the symmetric matrix of edge weights. This follows immediately, since $1 - x_i x_j = 0$ iff the vertices i and j are in the same set. The objective can be written as

$$\frac{1}{2} \sum_{i < j} a_{ij} (1 - x_i x_j) = \frac{1}{4} x^T (\text{Diag}(Ae) - A) x = \frac{1}{4} x^T L x,$$

where $L := \text{Diag}(Ae) - A$ is the *Laplace matrix* of G . Using $x^T L x = \text{trace}(L(x x^T))$ and $X = x x^T$, it can be shown that this problem is equivalent to

$$\hat{f}_p = \min \frac{1}{4} \langle L, X \rangle \quad \text{subject to} \quad \text{diag}(X) = e, \quad e^T X e = 0, \quad X \succeq 0, \quad \text{rank}(X) = 1.$$

Since $X \succeq 0$ and $e^T X e = 0$ implies X to be singular, the problem is ill-posed, and for arbitrarily small perturbations of the right hand side the problem becomes infeasible. By definition, the Equicut Problem has a finite optimal value \hat{f}_p , and a rigorous upper bound of \hat{f}_p is simply obtained by evaluating the objective function for a given partitioning integer vector x . Hence, it is left over to compute a rigorous lower bound. At first, the nonlinear rank one constraint is left out yielding an ill-posed semidefinite relaxation, where the Slater condition does not hold. The related constraints $\text{diag}(X) = e$ and $e^T X e = 0$ can be written as

$$\langle A_i, X \rangle = b_i, \quad b_i = 1, \quad A_i = E_i \text{ for } i = 1, \dots, n, \quad \text{and } A_{n+1} = e e^T, \quad b_{n+1} = 0.$$

where E_i is the $n \times n$ matrix with a one on the i th diagonal position and zeros otherwise. Hence, the dual semidefinite problem has the form

$$\max \sum_{i=1}^n y_i \quad \text{s.t.} \quad \text{diag}(y_{1:n}) + y_{n+1}(e e^T) \preceq \frac{1}{4} L, \quad y \in \mathbf{R}^{n+1}.$$

The constraints $\text{diag}(X) = e$, $X \succeq 0$ imply PBQ with finite upper bounds $\lambda_{\max}(X) \leq \bar{x} = n$. Corollary 6.1 yields

Corollary 9.1 *Let $\tilde{y} \in \mathbf{R}^{n+1}$, and assume that the matrix*

$$D = \frac{1}{4} L - \text{Diag}(\tilde{y}_{1:n}) - \tilde{y}_{n+1}(e e^T)$$

has at most l negative eigenvalues, and let $\underline{d} \leq \lambda_{\min}(D)$. Then

$$\hat{f}_p \geq \sum_{i=1}^n \tilde{y}_i + l \cdot n \cdot \underline{d} =: \underline{f}_p.$$

n	\underline{f}_p	$\mu(\widetilde{f}_p, \widetilde{f}_d)$	$\mu(\widetilde{f}_d, \underline{f}_p)$	t	t_{low}	tc
100	-3.58065e+003	-7.117e-008	3.843e-011	4.2	0.5	0
200	-1.04285e+004	-7.018e-008	9.621e-010	7.9	0.2	0
300	-1.90966e+004	-2.573e-008	8.918e-009	21.1	0.9	0
400	-3.01393e+004	-1.633e-008	3.008e-008	39.0	2.0	0
500	-4.22850e+004	1.431e-008	2.584e-008	67.5	3.7	0
600	-5.57876e+004	5.418e-009	1.829e-008	124.7	6.0	-5

Table 1: Results for Graph Partitioning

In Table 1 some numerical results for problems given by Gruber and Rendl [11] are displayed. The number of nodes is denoted by n . For this suite of ill-posed problems with up to 601 constraints and 180300 variables the semidefinite programming solver SDPT3, version 3.02 [46] has computed approximations of the dual optimal value \widetilde{f}_d , which are close to the approximate primal optimal value \widetilde{f}_p ; see the column $\mu(\widetilde{f}_p, \widetilde{f}_d)$. Here, the relative accuracy of two real numbers a and b is measured by the quantity

$$\mu(a, b) := \frac{a - b}{\max\{1.0, (|a| + |b|)/2\}}.$$

The negative signs in this column show that weak duality is violated for the computed approximations in four cases. SDPT3 gave $tc = 0$ (normal termination) for the first five ill-posed examples. Only in the last case $n = 600$ the warning $tc = -5$ (that means : *Progress too slow*) was returned. We have computed the lower bound \underline{f}_p by using Corollary 9.1. The small quantities $\mu(\widetilde{f}_d, \underline{f}_p)$ show that the overestimation of the rigorous lower bound \underline{f}_p can be neglected. In Table 1 the times for computing the approximations with SDPT3, and for computing \underline{f}_p by using Corollary 9.1 are denoted by t and t_{low} , respectively. It follows that the additional time t_{low} for computing the rigorous bound \underline{f}_p is small compared to the time t needed for the approximations. This is of the same tenor as the quotation of Turing at first.

10 Numerical Results

In this section we describe briefly our numerical experience. Lurupa [17] is a C++ implementation of the presented rigorous bounds for the special case of linear programming. In the following we give a short summary of numerical results for the NETLIB suite of linear programming problems [26]. For details refer to [19]. The NETLIB LP-suite is a well-known collection of difficult to solve problems with up to 15695 variables and 16675 constraints. They originate from various applications, for example forestry, flap settings on aircraft, and staff scheduling. We chose the set of problems for which Ordóñez and Freund [33] have

computed condition numbers. There it is stated that 71% of the problems have an infinite condition number. As Fourer and Gay [7] observed, preprocessing can change the state of an LP from feasible to infeasible and vice versa, and therefore preprocessing was not applied.

Roughly speaking, a finite lower bound (upper bound) of the optimal value can be computed iff the distance to dual infeasibility (primal infeasibility) is greater than zero. For 76 problems a finite lower bound could be computed with a median accuracy of $\text{med}(\mu(\tilde{f}, \bar{f}_p)) = 2.2 \cdot 10^{-8}$ (\tilde{f} is the approximate optimal value) and a median time ratio of $\text{med}(t_{\text{low}}/t) = 0.5$. For 35 problems Lurupa has computed a finite upper bound with $\text{med}(\mu(\tilde{f}, \bar{f}_d)) = 8.0 \cdot 10^{-9}$ and $\text{med}(t_{\text{up}}/t) = 5.3$. For 32 well-posed problems finite rigorous lower and upper bounds could be computed with $\text{med}(\mu(\bar{f}_p, \underline{f}_d)) = 5.6 \cdot 10^{-8}$. Only for two problems with finite condition number (SCSD8 and SCTAP1) an upper bound of the optimal value could not be computed. Taking into account the approximate solver's default stopping tolerance of 10^{-9} , the guaranteed accuracy computed with Lurupa for the NETLIB LP suite is reasonable. The upper bound is more expensive, since linear systems have to be solved rigorously, and sometimes perturbed problems have .

For the SDPLIB benchmark problems of Borchers [4] we have computed with VSDP [15], a MATLAB software package for verified semidefinite programming, rigorous bounds . For details see [15] and [16]. Freund, Ordóñez and Toh [8] have solved 85 problems with SDPT3 out of the 92 problems of the SDPLIB. They have omitted the four infeasible problems and three very large problems where SDPT3 produced out of memory. In their paper interior-point iteration counts with respect to different measures for semidefinite programming problems are investigated, and it is pointed out that 32 are ill-posed. VSDP could compute (by using SDPT3 as approximate solver) for all 85 problems a rigorous lower bound of the optimal value and could verify the existence of strictly dual feasible solutions, which proves that all problems have a zero duality gap. A finite rigorous upper bound could be computed for all well-posed problems with one exception; this is `hinf2`. For all 32 ill-posed problems VSDP has computed the upper bound $\bar{f}_d = +\infty$, which reflects exactly that the distance to the next primal infeasible problem is zero as well as the infinite condition number.

For the 85 test problems (not counting the 4 infeasible ones) SDPT3 (with default values) has computed good approximations and gave 7 warnings, and 2 warnings are given for well-posed problems. Hence, no warnings are given for 27 ill-posed problems with zero distance to primal infeasibility. In other words, there is no correlation between warnings and the difficulty of the problem. At least for this test set our rigorous bounds reflect the difficulty of the problems much better, and they provide safety, especially in the case where algorithms subsequently call other algorithms, as is done for example in branch-and-bound methods.

Some major characteristics of the numerical results of VSDP for the well-posed SDPLIB-problems are as follows: The median of the time ratio for computing the rigorous lower

(upper) bound and the approximation is 0.085, (1.99), respectively. The median of the guaranteed accuracy for the problems with finite condition number is $7.01 \cdot 10^{-7}$. We have used the median here because there are some outliers.

One of the largest problems which could be rigorously solved by VSDP is **thetaG51** where the number of constraints is $m = 6910$, and the dimension of the primal symmetric matrix X is $s = 1001$ (implying 501501 variables). For this problem SDPT3 gave the message out of memory, and we used SDPA [9] as approximate solver. The rigorous lower and upper bounds computed by VSDP are $\underline{f}_p = -3.4900 \cdot 10^2$, $\overline{f}_d = -3.4406 \cdot 10^2$, respectively. This is an outlier because the guaranteed relative accuracy is only 0.014, which may be sufficient in several applications, but is insufficient from a numerical point of view. However, existence of optimal solutions and strong duality is proved. The times in seconds for computing the approximations, the lower and the upper bound of the optimal value are $\mathfrak{t} = 3687.95$, $t_{low} = 45.17$, and $t_{up} = 6592.52$, respectively.

For further numerical results and applications concerning ill-posed problems and the problem of computing the ground state energy of atomic and molecular systems by using a variational approach (see for example Fukuda et al. [10] and Nakata et al. [23]) refer to VSDP [15].

To our knowledge no other software packages compute rigorous results for semidefinite programs. There are several packages that compute verified results for optimization problems where the objective and the constraints are defined by smooth algebraic expressions. Elaborate comparisons with some of these packages in the case of linear programming problems can be found in the forthcoming paper of Keil [18].

11 Conclusions

The computation of rigorous error bounds for conic optimization problems can be viewed as a carefully postprocessing tool that uses only approximate solutions computed by any conic solver. The bounds are developed in the framework of functional analysis. Error bounds for special conic problems can be derived easily.

Several numerical results demonstrate that rigorous error bounds can be reasonably easily computed even for problems of large size and for ill-conditioned problems, in most cases with a range which is not much larger than necessary.

References

- [1] G. Alefeld and J. Herzberger. *Introduction to Interval Computations*. Academic Press, New York, 1983.

- [2] F. Alizadeh and D. Glodfarb. Second-Order Cone Programming. *Math. Program.*, 95:3–51, 2003.
- [3] G.D. Birkhoff. Lattice theory, revised edition. In *Am. Math. Soc. Colloquium Publications*, volume 25. Am. Math. Soc. New York, 1948.
- [4] B. Borchers. SDPLIB 1.2, A Library of Semidefinite Programming Test Problems. *Optimization Methods and Software*, 11(1):683–690, 1999.
- [5] N. Bourbaki. *Éléments de mathématique. XIII. 1. part: Les structures fondamentales de l'analyse. Livre VI: Intégration.* Actualités scientifique et industrielles, 1952.
- [6] C.A. Floudas. *Deterministic Global Optimization - Theory, Methods and Applications*, volume 37 of *Nonconvex Optimization and Its Applications*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
- [7] R. Fourer and D. M. Gay. Experience with a primal presolve algorithm. In W. W. Hager, D. W. Hearn, and P. M. Pardalos, editors, *Large Scale Optimization: State of the Art*, pages 135–154. Kluwer Academic Publishers Group, Norwell, MA, USA, and Dordrecht, The Netherlands, 1994.
- [8] R.M. Freund, F. Ordóñez, and K. Toh. Behavioral Measures and their Correlation with IPM Iteration Counts on Semi-Definite Programming Problems. *Math. Programming*, 109(2):445–475, 2007.
- [9] K. Fujisawa, Y. Futakata, M. Kojima, K. Nakata, and M. Yamashita. SDPA-M (SemiDefinite Programming Algorithm in MATLAB) User's Manual — Version 2.00. Research Report B-359, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152-0033, Japan, 2000. Revised: July 2003.
- [10] M. Fukuda, B.J. Braams, M. Nakata, M.L. Overton, J.K. Percus, M. Yamashita, and Z. Zhao. Large-scale semidefinite programs in electronic structure calculation. *Mathematical Programming*, to appear.
- [11] G. Gruber and F. Rendl. Computational experience with ill-posed problems in semidefinite programming. *Computational Optimization and Applications*, 21(2):201–212, 2002.
- [12] ILOG CPLEX 7.1, User's manual. *ILOG*. France, 2001.
- [13] C. Jansson. A rigorous lower bound for the optimal value of convex optimization problems. *J. Global Optimization*, 28:121–137, 2004.
- [14] C. Jansson. Rigorous Lower and Upper Bounds in Linear Programming. *SIAM J. Optimization (SIOPT)*, 14(3):914–935, 2004.

- [15] C. Jansson. VSDP: A MATLAB software package for Verified Semidefinite Programming. *NOLTA*, pages 327–330, 2006.
- [16] C. Jansson, D. Chaykin, and C. Keil. Rigorous Error Bounds for the Optimal Value in Semidefinite Programming. *To appear in SIAM Journal on Numerical Analysis (SINUM)*, 2005.
- [17] C. Keil. Lurupa – Rigorous Error Bounds in Linear Programming. In B. Buchberger, S. Oishi, M. Plum, and S. M. Rump, editors, *Algebraic and Numerical Algorithms and Computer-assisted Proofs*, number 05391 in Dagstuhl Seminar Proceedings. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2006. <http://drops.dagstuhl.de/opus/volltexte/2006/445>.
- [18] C. Keil. Verified Linear Programming – a Comparison. Submitted, 2007.
- [19] C. Keil and C. Jansson. Computational Experience with Rigorous Error Bounds for the Netlib Linear Programming Library. *Reliable Computing*, 12, issue 4:303–321, 2006.
- [20] M. Laurent and S. Poljak. On a positive semidefinite relaxation of the cut polytope. *Linear Algebra and its Applications (LAA)*, 223/224:439–461, 1995.
- [21] G. Mayer. Result verification for eigenvectors and eigenvalues. In J. Herzberger, editor, *Topics in validated computations. Proceedings of the IMACS-GAMM international workshop, Oldenburg, Germany, 30 August - 3 September 1993*, Stud. Comput. Math. 5, pages 209–276, Amsterdam, 1994. Elsevier.
- [22] R.E. Moore. *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, 1979.
- [23] M. Nakata, M. Fukuda, K. Nakata, and K. Fujisawa. Variational calculus of fermion second-order reduced density matrices by semidefinite programming algorithm. *J. of Chemical Physics*, 114(19):8282–8292, 2001.
- [24] A. Nemirovskii. Interior Point Polynomial Time Methods in Convex Programming. Lecture Notes Faculty of Industrial Engineering and Management, Technion Israel Institute of Technology, Technion City, Haifa 32000, Israel, 1996. <http://iew3.technion.ac.il/Labs/Opt/index.php?4>.
- [25] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM, Philadelphia, 1994.
- [26] NETLIB Linear Programming Library.
- [27] A. Neumaier. *Interval Methods for Systems of Equations*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.

- [28] A. Neumaier. *Introduction to Numerical Analysis*. Cambridge University Press, 2001.
- [29] A. Neumaier. Complete Search in Continuous Global Optimization and Constraint Satisfaction. *Acta Numerica*, 13:271–369, 2004.
- [30] A. Neumaier. Computer-assisted proofs. Proc. IEEE SCAN, to appear, 2006.
- [31] A. Neumaier and O. Shcherbina. Safe bounds in linear and mixed-integer programming. *Mathematical Programming, Ser. A*, 99:283–296, 2004.
- [32] J.v. Neumann and H.H. Goldstine. Numerical Inverting of Matrices of High Order. *Bull. Amer. Math. Soc.* 53, pages 1021–1099, 1947.
- [33] F. Ordóñez and R.M. Freund. Computational experience and the explanatory value of condition measures for linear optimization. *SIAM J. Optimization (SIOPT)*, 14(2):307–333, 2003.
- [34] B.N. Parlett. The Accuracy Issue and Numerical Eigenproblems. *SIAM News*, 34(1):4 pages, 2001.
- [35] A.L. Peressini. *Ordered Topological Vector Spaces*. Harper and Row, 1967.
- [36] J. Renegar. Some perturbation theory for linear programming. *Mathematical Programming*, 65(1(A)):79–91, 1994.
- [37] J. Renegar. Linear Programming, complexity theory, and elementary functional analysis. *Mathematical Programming*, 70(3):279–351, 1995.
- [38] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.
- [39] S.M. Rump. Validated Solution of Large Linear Systems. In R. Albrecht, G. Alefeld, and H.J. Stetter, editors, *Validation numerics: theory and applications*, volume 9 of *Computing Supplementum*, pages 191–212. Springer, 1993.
- [40] S.M. Rump. Verification Methods for Dense and Sparse Systems of Equations. In J. Herzberger, editor, *Topics in Validated Computations — Studies in Computational Mathematics*, pages 63–136, Elsevier, Amsterdam, 1994.
- [41] S.M. Rump. Computer-assisted proofs and Self-Validating Methods. In B. Einarsson, editor, *Handbook on Accuracy and Reliability in Scientific Computation*, pages 195–240. SIAM, 2005.
- [42] S.M. Rump. INTLAB - Interval Laboratory, the Matlab toolbox for verified computations, Version 5.3, 2006.
- [43] H.H. Schaefer. *Banach lattices and positive operators*. Springer, 1974.

- [44] L.N. Trefethen. Numerical analysis. Technical Report 06/06, Oxford University Computing Laboratory, 2006.
<http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/NAessay.pdf>.
- [45] A. M. Turing. Rounding-Off Errors in Matrix Processes. *Quarterly J. of Mechanics & App. Maths.*, 1(1):287–308, 1948.
- [46] R.H. Tütüncü, K.C. Toh, and M.J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Math. Program.*, 95B(2):189–217, 2003.
- [47] H. Wolkowicz. Semidefinite and Cone Programming Bibliography, Comments.
<http://orion.uwaterloo.ca/~hwolkowi/henry/book/fronthandbk.d/sdpbibliog.pdf>.